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A singular boundary value problem for odd-order differential equations ☆

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Abstract

The odd-order differential equation $(-1)^n x^{(2n+1)} = f(t, x, \dots, x^{(2n)})$ together with the Lidstone boundary conditions $x^{(2j)}(0) = x^{(2j)}(T) = 0$, $0 \leqslant j \leqslant n-1$, and the next condition $x^{(2n)}(0) = 0$ is discussed. Here f satisfying the local Carathéodory conditions can have singularities at the value zero of all its phase variables. Existence result for the above problem is proved by the general existence principle for singular boundary value problems.

Keywords: Odd-order differential equation; Singular boundary value problem; Existence; Regularization

1. Introduction

Let T be a positive number, $J = [0, T]$, $\mathbb{R}_- = (-\infty, 0)$, $\mathbb{R}_+ = (0, \infty)$ and $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$. Let $n \in \mathbb{N}$ and $\mathcal{D}_n \subset \mathbb{R}^{2n+1}$ be defined by

$$\mathcal{D}_n = \begin{cases} \underbrace{\mathbb{R}_+ \times \mathbb{R}_0 \times \mathbb{R}_- \times \mathbb{R}_0 + \times \cdots \times \mathbb{R}_-}_{4k-1} & \text{if } n = 2k-1, \\ \underbrace{\mathbb{R}_+ \times \mathbb{R}_0 \times \mathbb{R}_- \times \mathbb{R}_0 + \times \cdots \times \mathbb{R}_+}_{4k+1} & \text{if } n = 2k \end{cases}$$

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(for $n=1$ and $n=2$ we have $\mathcal{D}_1=\mathbb{R}_+\times\mathbb{R}_0\times\mathbb{R}_-$ and $\mathcal{D}_2=\mathbb{R}_+\times\mathbb{R}_0\times\mathbb{R}_-\times\mathbb{R}_0\times\mathbb{R}_+$, respectively.

Consider the singular boundary value problem (BVP)

$$(-1)^n x^{(2n+1)}(t)=f\big(t,x(t),\dots,x^{(2n)}(t)\big), \tag{1.1}$$

$$x^{(2n)}(0)=0, \qquad x^{(2j)}(0)=x^{(2j)}(T)=0, \quad 0\leqslant j\leqslant n-1, \tag{1.2}$$

where positive f ($\in \operatorname{Car}(J\times\mathcal{D}_n)$) satisfying the local Carathéodory conditions on $J\times\mathcal{D}_n$ may be singular at the value 0 of all its phase variables in the following sense: $\lim_{x_j\to 0} f(t,x_0,\dots,x_j,\dots,x_{2n})=\infty$ for $0\leqslant j\leqslant 2n$, a.e. $t\in J$ and each $(x_0,\dots,x_{j-1},x_{j+1},\dots,x_{2n})$ such that $(x_0,\dots,x_{j-1},x_j,x_{j+1},\dots,x_{2n})\in\mathcal{D}_n$.

We say that a function $x\in AC^{2n}(J)$ (the set of functions whose $2n$-derivatives are absolutely continuous on J) is *a solution of* BVP (1.1), (1.2) if x satisfies the boundary conditions (1.2) and for a.e. $t\in J$ fulfills (1.1).

We recall that the BVP

$$(-1)^n x^{(2n)}(t)=f_1\big(t,x(t),\dots,x^{(2n-1)}(t)\big), \tag{1.3}$$

$$x^{(2j)}(0)=x^{(2j)}(T)=0, \quad 0\leqslant j\leqslant n-1, \tag{1.4}$$

is the well-known Lidstone BVP which the special case (for $n=1$) is the Dirichlet BVP. Here the differential equation (1.3) has an even order. The Lidstone BVP (with general n) have been studied in the regular case, e.g., by Agarwal [1], Agarwal and Wong [9,10], and Palamides [7]; and in the singular case by Agarwal, O'Regan, Rachůnková, and Staněk [4]. In this paper we consider the differential equation (1.1) of an odd order together with the Lidstone boundary conditions (1.4) and the next condition $x^{(2n)}(0)=0$. So, we can keep looking at our BVP (1.1), (1.2) as an 'expansion' of the Lidstone BVP.

The aim of this paper is to give conditions guaranteeing the solvability of BVP (1.1), (1.2). Our approach is based on the general existence principle which was proved by the authors in [8].

Throughout the paper, $\|x\|=\max\{|x(t)|: t\in J\}$, $\|x\|_{C^{2n}}=\sum_{j=0}^{2n}\|x^{(j)}\|$, and $\|x\|_{L_1}=\int_0^T|x(t)|\,dt$ stands for the norm in $C^0(J)$, $C^{(2n)}(J)$, and $L_1(J)$, respectively. For each measurable $\mathcal{M}\subset\mathbb{R}$, $\mu(\mathcal{M})$ denotes its Lebesgue measure.

Throughout the paper the following assumptions are used:[1]

(H_1) $f\in\operatorname{Car}(J\times\mathcal{D}_n)$ and there exist $S\in\mathbb{R}_+$ and $r\in[0,1)$ such that

$$\frac{S}{t^r}\leqslant f(t,x_0,\dots,x_{2n})$$

for a.e. $t\in J$ and each $(x_0,\dots,x_{2n})\in\mathcal{D}_n$.

(H_2) For a.e. $t\in J$ and each $(x_0,\dots,x_{2n})\in\mathcal{D}_n$,

$$f(t,x_0,\dots,x_{2n})\leqslant h\left(t,\sum_{j=0}^{2n}|x_j|\right)+\sum_{j=0}^{n}\omega_{2j}\big(|x_{2j}|\big)+\sum_{j=0}^{n-1}\omega_{2j+1}\big(|x_{2j+1}|\big)$$

[1] In the sequel, if some conditions or statements depend on $j\in\mathbb{N}$ with $0\leqslant j\leqslant n-2$, then those come into force only if $n\geqslant 2$.

where $h \in \mathrm{Car}(J \times [0,\infty))$ is positive and non-decreasing in the second variable, $\omega_j \in C^0(\mathbb{R}_+)$ $(0 \leqslant j \leqslant 2n)$ are positive and non-increasing,

$$\limsup_{u\to\infty} \frac{1}{u} \int_0^T h(t, Vu)\,dt < 1 \tag{1.5}$$

with

$$V = \begin{cases} 2n+1 & \text{if } T = 1, \\ \dfrac{T^{2n+1}-1}{T-1} & \text{if } T \neq 1, \end{cases} \tag{1.6}$$

$$\int_0^1 \omega_{2j}(s)\,ds < \infty \quad \text{for } 0 \leqslant j \leqslant n-1,$$

$$\int_0^1 \omega_{2j+1}\big(s^2\big)\,ds < \infty \quad \text{for } 0 \leqslant j \leqslant n-2, \tag{1.7}$$

$$\int_0^1 \omega_{2n-1}\big(s^{2-r}\big)\,ds < \infty, \qquad \int_0^1 \omega_{2n}\big(s^{1-r}\big)\,ds < \infty. \tag{1.8}$$

Remark 1.1. Since $\omega_j : \mathbb{R}_+ \to \mathbb{R}_+$ $(0 \leqslant j \leqslant 2n)$ in (H_2) are continuous and non-increasing, the assumptions (1.7) and (1.8) imply that all these integrals are convergent also in the case when we write $c \in \mathbb{R}_+$ instead of 1 in their upper bounds.

As we have written, our existence result for BVP (1.1), (1.2) is proved by the general existence principle. To formulate that, consider an auxiliary sequence of regular differential equations

$$(-1)^n x^{(2n+1)}(t) = f_m\big(t, x(t), \ldots, x^{(2n)}(t)\big), \tag{1.9}$$

where $f_m \in \mathrm{Car}(J \times \mathbb{R}^{2n+1})$, $m \in \mathbb{N}$, together with a general boundary condition of the form

$$x \in \mathcal{B}, \tag{1.10}$$

where $\mathcal{B}$ is a closed subset in $C^{2n}(J)$. Then the general existence principle of [8] for singular BVP (1.1), (1.10) is given in the following theorem.

Theorem 1.2 (General existence principle) [6, Theorem 1.3]. *Let us suppose that there is a bounded set $\Omega \subset C^{2n}(J)$ such that*

(i) *for each $m \in \mathbb{N}$, the (regular) BVP* (1.9), (1.10) *has a solution $x_m \in \Omega$,*

(ii) *the sequence* $\{f_m(t, x_m(t), \dots, x_m^{(2n)}(t))\}$ *is uniformly absolutely continuous on* J, *that is for each* $\varepsilon > 0$ *there is a* $\delta > 0$ *such that*

$$\int\limits_{\mathcal{M}} f_m\big(t, x_m(t), \dots, x_m^{(2n)}(t)\big)\, dt < \varepsilon, \quad m \in \mathbb{N},$$

whenever $\mathcal{M} \subset J$ *is measurable and* $\mu(\mathcal{M}) < \delta$.

Then we have

(I) *there exist* $x \in \overline{\Omega}$ *and a subsequence* $\{x_{k_m}\}$ *of* $\{x_m\}$ *such that*

$$\lim_{m\to\infty} \|x_{k_m} - x\|_{C^{2n}} = 0, \tag{1.11}$$

(II) *if for a.e.* $t \in J$

$$\lim_{m\to\infty} f_{k_m}\big(t, x_{k_m}(t), \dots, x_{k_m}^{(2n)}(t)\big) = f\big(t, x(t), \dots, x^{(2n)}(t)\big), \tag{1.12}$$

then x *is a solution of the singular BVP* (1.1), (1.10).

In the paper we assume that $f \in \mathrm{Car}(J \times \mathcal{D}_n)$ and so it may be singular at the value 0 of all its phase variables, only. In this case Theorem 1.2 leads to:

Theorem 1.3 (Existence principle for BVPs with phase singularities at 0). *Let* $f_m \in \mathrm{Car}(J \times \mathbb{R}^{2n+1})$, $m \in \mathbb{N}$. *Assume that for a.e.* $t \in J$ *and all* $m \in \mathbb{N}$, $(x_0, \dots, x_{2n}) \in \mathcal{D}_n$,

$$f_m(t, x_0, \dots, x_{2n}) = f(t, x_0, \dots, x_{2n}) \quad \text{if } |x_i| \geqslant \frac{1}{m},\ 0 \leqslant i \leqslant 2n. \tag{1.13}$$

Assume also that there is a bounded set $\Omega \subset C^{2n}(J)$ *such that*

(a) *for each* $m \in \mathbb{N}$, *the (regular) BVP* (1.9), (1.10) *has a solution* $x_m \in \Omega$,
(b) *for each* $\varepsilon > 0$ *there is a* $\delta > 0$ *such that if* $\{(a_k, b_k)\}_{k\in\mathbb{J}}$ *is an at most countable set of mutually disjoint intervals* $(a_k, b_k) \subset (0, T)$ *and if* $\sum_{k\in\mathbb{J}}(b_k - a_k) < \delta$, *then*

$$\sum_{j\in\mathbb{J}} \int\limits_{a_k}^{b_k} \big|f_m\big(t, x_m(t), \dots, x_m^{(2n)}(t)\big)\big|\, dt < \varepsilon, \quad m \in \mathbb{N}.$$

Then we have

(A) *there exist* $x \in \overline{\Omega}$ *and a subsequence* $\{x_{k_m}\}$ *of* $\{x_m\}$ *satisfying* (1.11),
(B) *if* $\mu(\mathcal{V}) = 0$, *where* $\mathcal{V}$ *is the set of all zeros of the functions* $x^{(i)}$ *with* $0 \leqslant i \leqslant 2n$, *then* x *is a solution of the singular BVP* (1.1), (1.10).

Proof. We see that conditions (i) and (ii) of Theorem 1.2 are satisfied. Really, condition (i) coincides with (a) and condition (ii) is equivalent to (b) by Remark 1.4 in [8]. Therefore assertion (A) is true. Now, we prove assertion (B). Let $\mathcal{V} \subset J$ be the set of all zeros of the functions $x^{(i)}$, $0 \leqslant i \leqslant 2n$, and let $\mu(\mathcal{V}) = 0$. Next, let $\mathcal{U}_1 \subset J$ be the set of all $t \in J$ such that $f(t, \cdot, \ldots, \cdot) : \mathcal{D}_n \to \mathbb{R}$ is not continuous. Since $f \in \mathrm{Car}(J \times \mathcal{D}_n)$, we have $\mu(\mathcal{U}_1) = 0$. Finally, let $\mathcal{U}_2 \subset J$ be the set of all $t \in J$ such that (1.13) falls. Them $\mu(\mathcal{U}_2) = 0$. Choose an arbitrary fixed $t_0 \in J \setminus (\mathcal{V} \cup \mathcal{U}_1 \cup \mathcal{U}_2)$. Then $|x^{(i)}(t_0)| > 0$ for $0 \leqslant i \leqslant 2n$ and, by (1.11), there exists $m_0 \in \mathbb{N}$ such that for each $m \geqslant m_0$

$$\left|x^{(i)}(t_0)\right| > \frac{1}{k_{m_0}}, \quad \left|x_{k_m}^{(i)}(t_0)\right| \geqslant \frac{1}{k_{m_0}}, \quad 0 \leqslant i \leqslant 2n.$$

Therefore, by (1.13), if $m \geqslant m_0$ then

$$f_{k_m}\bigl(t_0, x_{k_m}(t_0), \ldots, x_{k_m}^{(2n)}(t_0)\bigr) = f\bigl(t_0, x_{k_m}(t_0), \ldots, x_{k_m}^{(2n)}(t_0)\bigr).$$

Since $t_0 \notin \mathcal{U}_1$, we have

$$\lim_{m\to\infty} \left| f_{k_m}\bigl(t_0, x_{k_m}(t_0), \ldots, x_{k_m}^{(2n)}(t_0)\bigr) - f\bigl(t_0, x(t_0), \ldots, x^{(2n)}(t_0)\bigr)\right|$$

$$= \lim_{m\to\infty} \left| f\bigl(t_0, x_{k_m}(t_0), \ldots, x_{k_m}^{(2n)}(t_0)\bigr) - f\bigl(t_0, x(t_0), \ldots, x^{(2n)}(t_0)\bigr)\right| = 0.$$

Hence, for a.e. $t \in J$ the condition (1.12) is fulfilled and so, by assertion (II) of Theorem 1.2, x is a solution of BVP (1.1), (1.10). □

The paper is organized as follows. In Section 2 we present some properties of the Green's function $G_j(t,s)$ for the problem $x^{(2j)}(t) = 0$, $x^{(2i)}(0) = x^{(2i)}(T) = 0$, $0 \leqslant i \leqslant j-1$. Section 3 deals with a sequence of auxiliary regular BVPs to problem (1.1), (1.2) where nonlinearities f_m in the regular differential equations satisfy conditions (1.13). We give a priori bounds for their solutions x_m and prove their existence by the topological transversality principle (see, e.g., [2,5,6]). In addition, we prove that the sequence $\{f_m(t, x_m(t), \ldots, x_m^{(2n)}(t))\}$ is uniformly absolutely continuous on J. In Section 4, applying a modification of general existence principle for singular BVPs given in Theorem 1.3, the existence result to BVP (1.1), (1.2) is proved.

2. Lemmas

Given $j \in \mathbb{N}$. From now on, $G_j(t,s)$ denotes the Green's function of the BVP

$$x^{(2j)}(t) = 0, \qquad x^{(2i)}(0) = x^{(2i)}(T) = 0, \quad 0 \leqslant i \leqslant j-1.$$

Then

$$G_1(t,s) = \begin{cases} \dfrac{s}{T}(t-T) & \text{for } 0 \leqslant s \leqslant t \leqslant T, \\ \dfrac{t}{T}(s-T) & \text{for } 0 \leqslant t < s \leqslant T, \end{cases}$$

and $G_j(t,s)$ can be expressed as [1,3,9]

$$G_j(t,s) = \int_0^T G_1(t,u)\, G_{j-1}(u,s)\, du, \quad j > 1.$$

It is known that [3,9]

$$(-1)^j G_j(t,s) > 0 \quad \text{for } (t,s) \in (0,T) \times (0,T). \tag{2.1}$$

Lemma 2.1 [4]. *For $(t,s) \in J \times J$ and $j \in \mathbb{N}$, we have*

$$(-1)^j G_j(t,s) \leqslant \frac{T^{2j-3}}{6^{j-1}} s(T-s),$$
$$(-1)^j G_j(t,s) \geqslant \frac{T^{2j-5}}{30^{j-1}} st(T-s)(T-t).$$

Lemma 2.2. *For each $t, \xi \in J$, we have*

$$\left| \int_\xi^t s(T-s)\, ds \right| \geqslant \frac{T}{6}(t-\xi)^2. \tag{2.2}$$

Proof. Let $t, \xi \in J$. Then

$$\left| \int_\xi^t s(T-s)\, ds \right| = \frac{1}{6}|t-\xi| \left| 3T(t+\xi) - 2\left(t^2 + t\xi + \xi^2\right) \right|.$$

Since $3T(t+\xi) - 2(t^2 + t\xi + \xi^2) \geqslant T|t-\xi|$, we see that (2.2) is true. □

3. Auxiliary regular BVPs

Here we construct auxiliary regular functions f_m satisfying (1.13). For each $m \in \mathbb{N}$, define $\chi_m, \varphi_m, \tau_m \in C^0(\mathbb{R})$, $\mathbb{R}_m \subset \mathbb{R}$ and $f_m \in \mathrm{Car}(J \times \mathbb{R}^{2n+1})$ by the formulas

$$\chi_m(u) = \begin{cases} u & \text{for } u \geqslant \dfrac{1}{m}, \\ \dfrac{1}{m} & \text{for } u < \dfrac{1}{m}, \end{cases} \qquad \varphi_m(u) = \begin{cases} -\dfrac{1}{m} & \text{for } u > -\dfrac{1}{m}, \\ u & \text{for } u \leqslant -\dfrac{1}{m}, \end{cases}$$

$$\tau_m = \begin{cases} \chi_m & \text{if } n = 2k, \\ \varphi_m & \text{if } n = 2k-1, \end{cases} \qquad \mathbb{R}_m = \left(-\infty, -\frac{1}{m}\right] \cup \left[\frac{1}{m}, \infty\right)$$

and

$$
\begin{aligned}
&f_m(t, x_0, x_1, x_2, x_3, \ldots, x_{2n-1}, x_{2n})\\
&\quad= \begin{cases}
f\big(t, \chi_m(x_0), x_1, \varphi_m(x_2), x_3, \ldots, x_{2n-1}, \tau_m(x_{2n})\big)\\
\qquad \text{for } (t, x_0, x_1, x_2, x_3, \ldots, x_{2n-1}, x_{2n}) \in J \times \mathbb{R} \times \mathbb{R}_m \times \mathbb{R} \times \mathbb{R}_m \times \cdots\\
\qquad\qquad \times \mathbb{R}_m \times \mathbb{R},\\
\dfrac{m}{2}\Big[f_m\Big(t, x_0, \dfrac{1}{m}, x_2, x_3, \ldots, x_{2n-1}, x_{2n}\Big)\Big(x_1 + \dfrac{1}{m}\Big)\\
\qquad - f_m\Big(t, x_0, -\dfrac{1}{m}, x_2, x_3, \ldots, x_{2n-1}, x_{2n}\Big)\Big(x_1 - \dfrac{1}{m}\Big)\Big]\\
\qquad \text{for } (t, x_0, x_1, x_2, x_3, \ldots, x_{n-1}, x_{2n}) \in J \times \mathbb{R} \times \Big(-\dfrac{1}{m}, \dfrac{1}{m}\Big) \times \mathbb{R} \times \mathbb{R}_m\\
\qquad\qquad \times \cdots \times \mathbb{R}_m \times \mathbb{R},\\
\dfrac{m}{2}\Big[f_m\Big(t, x_0, x_1, x_2, \dfrac{1}{m}, \ldots, x_{2n-1}, x_{2n}\Big)\Big(x_3 + \dfrac{1}{m}\Big)\\
\qquad - f_m\Big(t, x_0, x_1, x_2, -\dfrac{1}{m}, \ldots, x_{2n-1}, x_{2n}\Big)\Big(x_3 - \dfrac{1}{m}\Big)\Big]\\
\qquad \text{for } (t, x_0, x_1, x_2, x_3, \ldots, x_{2n-1}, x_{2n}) \in J \times \mathbb{R}^3 \times \Big(-\dfrac{1}{m}, \dfrac{1}{m}\Big) \times \cdots\\
\qquad\qquad \times \mathbb{R}_m \times \mathbb{R},\\
\vdots\\
\dfrac{m}{2}\Big[f_m\Big(t, x_0, x_1, x_2, \ldots, \dfrac{1}{m}, x_{2n}\Big)\Big(x_{2n-1} + \dfrac{1}{m}\Big)\\
\qquad - f_m\Big(t, x_0, x_1, x_2, \ldots, -\dfrac{1}{m}, x_{2n}\Big)\Big(x_{2n-1} - \dfrac{1}{m}\Big)\Big]\\
\qquad \text{for } (t, x_0, x_1, x_2, x_3, \ldots, x_{2n-1}, x_{2n}) \in J \times \mathbb{R}^{2n-1} \times \Big(-\dfrac{1}{m}, \dfrac{1}{m}\Big) \times \mathbb{R}.
\end{cases}
\end{aligned}
$$

By (H_1) and (H_2),

$$
\begin{aligned}
\frac{S}{t^r} &\leqslant f_m(t, x_0, \ldots, x_{2n})\\
&\leqslant h\Bigg(t, 2n+1+\sum_{j=0}^{2n}|x_j|\Bigg) + \sum_{j=0}^{n}\omega_{2j}\big(|x_{2j}|\big) + \sum_{j=0}^{n-1}\omega_{2j+1}\big(|x_{2j+1}|\big)
\end{aligned}
\tag{3.1}
$$

for a.e. $t \in J$ and each $(x_0, \ldots, x_{2n}) \in \mathbb{R}_0^{2n+1}$. Also

$$
\begin{aligned}
\frac{S}{t^r} &\leqslant f_m(t, x_0, \ldots, x_{2n})\\
&\leqslant h\Bigg(t, 2n+1+\sum_{j=0}^{2n}|x_j|\Bigg) + \sum_{j=0}^{n}\big[\omega_{2j}\big(|x_{2j}|\big)\big]_m + \sum_{j=0}^{n-1}\big[\omega_{2j+1}\big(|x_{2j+1}|\big)\big]_m
\end{aligned}
\tag{3.2}
$$

for a.e. $t \in J$ and each $(x_0, \dots, x_{2n}) \in \mathbb{R}^{2n+1}$, where (for $0 \leqslant k \leqslant 2n$)

$$[\omega_k(u)]_m = \begin{cases} \omega_k(u) & \text{for } u \in \left[\dfrac{1}{m}, \infty\right), \\ \omega_k\left(\dfrac{1}{m}\right) & \text{for } u \in \left[0, \dfrac{1}{m}\right]. \end{cases} \tag{3.3}$$

To prove the solvability of the sequence of regular BVPs (1.9), (1.2), we first consider the two parameter family of regular differential equations

$$(-1)^n x^{(2n+1)}(t) = \lambda f_m\big(t, x(t), \dots, x^{(2n)}(t)\big) \tag{3.4}$$

depending on the parameters $\lambda \in [0,1]$ and $m \in \mathbb{N}$. A priori bounds for solutions of BVPs (3.4), (1.2) are given in the lemma.

Lemma 3.1. *Let assumptions* (H_1) *and* (H_2) *be satisfied and* $m \in \mathbb{N}$. *Let* x *be a solution of BVP* (3.4), (1.2) *for some* $\lambda \in [0,1]$. *Then there exists a positive constant* L_m *independent of* λ *such that*

$$\big\|x^{(j)}\big\| < T^{2n-j} L_m \quad \textit{for } 0 \leqslant j \leqslant 2n. \tag{3.5}$$

Proof. Since

$$(-1)^n x^{(2n)}(t) = \lambda \int_0^t f_m\big(s, x(s), \dots, x^{(2n)}(s)\big)\, ds, \quad t \in J, \tag{3.6}$$

we have (see (3.2))

$$\big|x^{(2n)}(t)\big| \geqslant \lambda S \int_0^t \frac{1}{s^r}\, ds = \frac{\lambda S}{1-r} t^{1-r}, \quad t \in J. \tag{3.7}$$

Now $x^{(2j)}(0) = x^{(2j)}(T) = 0$ $(0 \leqslant j \leqslant n-1)$ give

$$x^{(2j+1)}(\xi_{2j+1}) = 0 \quad \text{for } 0 \leqslant j \leqslant n-1, \tag{3.8}$$

where $\xi_{2j+1} \in (0,T)$ and then using (3.7) and $x^{(2n-1)}(\xi_{2n-1}) = 0$ we get

$$\begin{aligned} \big|x^{(2n-1)}(t)\big| &= \left| \int_{\xi_{2n-1}}^t x^{(2n)}(s)\, ds \right| \geqslant \frac{\lambda S}{1-r} \left| \int_{\xi_{2n-1}}^t s^{1-r}\, ds \right| \\ &= \frac{\lambda S}{(1-r)(2-r)} \big|t^{2-r} - \xi_{2n-1}^{2-r}\big|. \end{aligned}$$

Then from the inequality $|t^{2-r} - u^{2-r}| \geqslant |t-u|^{2-r}$ for $t, u \in J$ it follows that

$$\big|x^{(2n-1)}(t)\big| \geqslant \frac{\lambda S}{(1-r)(2-r)} |t - \xi_{2n-1}|^{2-r}, \quad t \in J. \tag{3.9}$$

Since

$$(-1)^j x^{(2j)}(t) = \int_0^T (-1)^{n-j} G_{n-j}(t,s)(-1)^n x^{(2n)}(s)\,ds, \quad t \in J,\ 0 \leqslant j \leqslant n-1,$$

Lemma 2.1, (2.1), (3.2) and (3.7) show that (for $t \in J$, $0 \leqslant j \leqslant n-1$)

$$\begin{aligned}(-1)^j x^{(2j)}(t) &= \int_0^T |G_{n-j}(t,s)|\bigl|x^{(2n)}(s)\bigr|\,ds \\ &\geqslant \frac{\lambda S T^{2(n-j)-5}}{(1-r)30^{n-j-1}} t(T-t) \int_0^T s^{2-r}(T-s)\,ds.\end{aligned}$$

Hence

$$(-1)^j x^{(2j)}(t) = \bigl|x^{(2j)}(t)\bigr| \geqslant \lambda H_{2j} t(T-t) \quad \text{for } t \in J,\ 0 \leqslant j \leqslant n-1, \tag{3.10}$$

where

$$H_{2j} = \frac{S T^{2(n-j)-5}}{(1-r)30^{n-j-1}} \int_0^T s^{2-r}(T-s)\,ds. \tag{3.11}$$

Then (see (3.8) and Lemma 2.2)

$$\begin{aligned}\bigl|x^{(2j+1)}(t)\bigr| &= \left|\int_{\xi_{2j+1}}^t x^{(2j+2)}(s)\,ds\right| \geqslant \lambda H_{2j+2}\left|\int_{\xi_{2j+1}}^t s(T-s)\,ds\right| \\ &\geqslant \frac{\lambda H_{2j+2} T}{6}(t-\xi_{2j+1})^2\end{aligned} \tag{3.12}$$

for $t \in J$, $0 \leqslant j \leqslant n-2$. Now from (3.2), (3.6), (3.7), (3.9), (3.10), (3.12) and using the inequalities (for $[\omega_k(u)]_m$ see (3.3))

$$\int_0^T \left[\omega_{2j+1}\left(\frac{\lambda H_{2j+2} T}{6}(s-\xi_{2j+1})^2\right)\right]_m ds < 2\int_0^T \left[\omega_{2j+1}\left(\frac{\lambda H_{2j+2} T}{6} s^2\right)\right]_m ds$$

for $0 \leqslant j \leqslant n-2$ and

$$\begin{aligned}&\int_0^T \left[\omega_{2n-1}\left(\frac{\lambda S}{(1-r)(2-r)}|s-\xi_{2n-1}|^{2-r}\right)\right]_m ds \\ &\quad < 2\int_0^T \left[\omega_{2n-1}\left(\frac{\lambda S}{(1-r)(2-r)} s^{2-r}\right)\right]_m ds,\end{aligned}$$

we have

$$\begin{aligned}
\left|x^{(2n)}(t)\right| &\leqslant \lambda \int_0^T \left[h\left(s, 2n+1+\sum_{j=0}^{2n}\left|x^{(j)}(s)\right|\right) + \sum_{j=0}^{n}\left[\omega_{2j}\left(\left|x^{(2j)}(s)\right|\right)\right]_m \right. \\
&\qquad \left. + \sum_{j=0}^{n-1}\left[\omega_{2j+1}\left(\left|x^{(2j+1)}(s)\right|\right)\right]_m \right] ds \\
&< \int_0^T h\left(s, 2n+1+\sum_{j=0}^{2n}\left\|x^{(j)}\right\|\right) + U_m(\lambda), \qquad (3.13)
\end{aligned}$$

where

$$\begin{aligned}
U_m(\varrho) = \varrho \Biggl\{ &\sum_{j=0}^{n-1}\int_0^T \left[\omega_{2j}\left(\varrho H_{2j} s(T-s)\right)\right]_m ds + \int_0^T \left[\omega_{2n}\left(\frac{\varrho S}{1-r}s^{1-r}\right)\right]_m ds \\
&+ 2\sum_{j=0}^{n-2}\int_0^T \left[\omega_{2j+1}\left(\frac{\varrho H_{2j+2}T}{6}s^2\right)\right]_m ds \\
&+ 2\int_0^T \left[\omega_{2n-1}\left(\frac{\varrho S}{(1-r)(2-r)}s^{2-r}\right)\right]_m ds \Biggr\} \qquad (3.14)
\end{aligned}$$

for $\varrho \in [0,1]$. Then $U_m \in C^0([0,1])$ by (1.7) and (1.8). Set

$$\Delta_m = \max\left\{U_m(\varrho):\ 0 \leqslant \varrho \leqslant 1\right\}. \qquad (3.15)$$

Further

$$\left\|x^{(j)}\right\| \leqslant T^{2n-j}\left\|x^{(2n)}\right\| \quad \text{for } 0 \leqslant j \leqslant 2n-1, \qquad (3.16)$$

which follows immediately using (3.8) and $x^{(2j)}(0) = 0$ $(0 \leqslant j \leqslant n-1)$. Hence (see (3.13)–(3.16))

$$\begin{aligned}
\left\|x^{(2n)}\right\| &\leqslant \int_0^T h\left(t, 2n+1+\left\|x^{(2n)}\right\|\sum_{j=0}^{2n}T^{2n-j}\right) dt + \Delta_m \\
&= \int_0^T h\left(t, 2n+1+V\left\|x^{(2n)}\right\|\right) dt + \Delta_m, \qquad (3.17)
\end{aligned}$$

where V is given by (1.6). From (1.5) it follows the existence of a positive constant L_m such that

$$\frac{1}{u}\int_0^T h(t, 2n+1+Vu)\, dt + \frac{\Delta_m}{u} < 1$$

for any $u \geqslant L_m$, and then (3.17) shows that $\|x^{(2n)}\| < L_m$. The inequalities (3.5) now follow from (3.16). □

Lemma 3.2. *Let assumptions* (H_1) *and* (H_2) *be satisfied. Then BVP* (1.9), (1.2) *has a solution for each* $m \in \mathbb{N}$.

Proof. Fix $m \in \mathbb{N}$. Let

$$\Omega_m = \left\{x\colon\, x \in C^{2n}(J),\, \left\|x^{(j)}\right\| < T^{2n-j} L_m \text{ for } 0 \leqslant j \leqslant 2n\right\}$$

where L_m is a positive constant from Lemma 3.1 and define the operator $\mathcal{F}_m : \overline{\Omega}_m \to C^{2n}(J)$ by

$$\mathcal{F}_m(x) = (-1)^n \int_0^T \left[G_n(t,s) \int_0^s f_m\big(v, x(v), \dots, x^{(2n)}(v)\big)\, dv \right] ds.$$

Since $f_m \in \mathrm{Car}(J \times \mathbb{R}^{2n+1})$ standard arguments guarantee that $\mathcal{F}_m$ is a compact operator. By Lemma 3.1, $x \neq \lambda \mathcal{F}(x)$ for $x \in \partial\Omega_m$ and $\lambda \in [0,1]$. Therefore $\mathcal{F}_m$ has a fixed point u in Ω_m by the topological transversality principle (see, e.g., [2,5,6]). From the definition of $\mathcal{F}_m$ we see that u is a solution of BVP (1.9), (1.2). □

Lemma 3.2 shows that BVP (1.9), (1.2) has a solution for each $m \in \mathbb{N}$. The next lemma guarantee a priori bound for solutions of BVP (1.9), (1.2) independent of m.

Lemma 3.3. *Let assumptions* (H_1) *and* (H_2) *be satisfied and let x be a solution of BVP* (1.9), (1.2) *for some* $m \in \mathbb{N}$. *Then there exists a positive constant P independent of m such that*

$$\left\|x^{(j)}\right\| < T^{2n-j} P \quad \textit{for } 0 \leqslant j \leqslant 2n. \tag{3.18}$$

Proof. Applying the same procedures as in the proof of Lemma 3.1 we see that formulas (3.6)–(3.12) hold with $\lambda = 1$ and then

$$\int_0^T \omega_{2j+1}\left(\frac{H_{2j+2}T}{6}(s-\xi_{2j+1})^2\right) ds < 2\int_0^T \omega_{2j+1}\left(\frac{H_{2j+2}T}{6} s^2\right) ds, \quad 0 \leqslant j \leqslant n-2,$$
$$\int_0^T \omega_{2n-1}\left(\frac{S}{(1-r)(2-r)}|s-\xi_{2n-1}|^{2-r}\right) ds < 2\int_0^T \omega_{2n-1}\left(\frac{S}{(1-r)(2-r)} s^{2-r}\right) ds.$$

Hence

$$\begin{aligned}\left|x^{(2n)}(t)\right| \leqslant \int_0^T \Bigg[& h\left(s, 2n+1+\sum_{j=0}^{2n}\left|x^{(j)}(s)\right|\right) + \sum_{j=0}^{n} \omega_{2j}\left(\left|x^{(2j)}(s)\right|\right) \\ & + \sum_{j=0}^{n-1} \omega_{2j+1}\left(\left|x^{(2j+1)}(s)\right|\right)\Bigg] ds\end{aligned}$$

$$< \int_0^T h\left(s, 2n+1+\sum_{j=0}^{2n}\|x^{(j)}\|\right) + \sum_{j=0}^{n-1}\int_0^T \omega_{2j}\big(H_{2j}s(T-s)\big)\,ds$$
$$+ \int_0^T \omega_{2n}\left(\frac{S}{1-r}s^{1-r}\right)ds + 2\sum_{j=0}^{n-2}\int_0^T \omega_{2j+1}\left(\frac{H_{2j+2}T}{6}s^2\right)ds$$
$$+ 2\int_0^T \omega_{2n-1}\left(\frac{S}{(1-r)(2-r)}s^{2-r}\right)ds. \tag{3.19}$$

Set

$$K = \sum_{j=0}^{n-1}\int_0^T \omega_{2j}\big(H_{2j}s(T-s)\big)\,ds + \int_0^T \omega_{2n}\left(\frac{S}{1-r}s^{1-r}\right)ds$$
$$+ 2\sum_{j=0}^{n-2}\int_0^T \omega_{2j+1}\left(\frac{H_{2j+2}T}{6}s^2\right)ds$$
$$+ 2\int_0^T \omega_{2n-1}\left(\frac{S}{(1-r)(2-r)}s^{2-r}\right)ds. \tag{3.20}$$

By (1.7) and (1.8), K is a positive constant independent of $m \in \mathbb{N}$. Thus (see (3.16), (3.19) and (3.20))

$$\|x^{(2n)}\| \leqslant \int_0^T h\left(t, 2n+1+\|x^{(2n)}\|\sum_{j=0}^{2n}T^{2n-j}\right)dt + K$$
$$= \int_0^T h\big(t, 2n+1+V\|x^{(2n)}\|\big)\,dt + K, \tag{3.21}$$

where V is given by (1.6). Using (1.5) we see that

$$\frac{1}{u}\int_0^T h(t, 2n+1+Vu)\,dt + \frac{K}{u} < 1$$

for any $u \geqslant P$ with a positive constant P. Then (3.21) shows that $\|x^{(2n)}\| < P$ and (3.18) follows from (3.16). $\square$

Lemma 3.4. *Let assumptions* (H_1) *and* (H_2) *be satisfied and, for* $m \in \mathbb{N}$*, let* x_m *be a solution of BVP* (1.9), (1.2)*. Then the sequence*

$$\left\{f_m\big(t, x_m(t), \ldots, x_m^{(2n)}(t)\big)\right\} \subset L_1(J)$$

is uniformly absolutely continuous (*UAC*) *on* J.

Proof. By Lemma 3.3, there exists a positive constant P such that

$$\big\|x_m^{(j)}\big\| < T^{2n-j}P \quad \text{for } 0 \leqslant j \leqslant 2n,\ m \in \mathbb{N} \tag{3.22}$$

and it follows from (1.2) that (see (3.8))

$$x_m^{(2j+1)}(\xi_{m,2j+1}) = 0, \quad 0 \leqslant j \leqslant n-1,\ m \in \mathbb{N},$$

where $\xi_{m,2j+1} \in (0,T)$. Next, from the proof of Lemma 3.3 it may be concluded that (see (3.7), (3.9), (3.10) and (3.12) with $\lambda = 1$)

$$\big|x_m^{(2n)}(t)\big| \geqslant \frac{S}{1-r}t^{1-r}, \quad t \in J,\ m \in \mathbb{N}, \tag{3.23}$$

$$\big|x_m^{(2n-1)}(t)\big| \geqslant \frac{S}{(1-r)(2-r)}|t-\xi_{m,2n-1}|^{2-r}, \quad t \in J,\ m \in \mathbb{N}, \tag{3.24}$$

$$\big|x_m^{(2j)}(t)\big| \geqslant H_{2j}t(T-t), \quad t \in J,\ 0 \leqslant j \leqslant n-1,\ m \in \mathbb{N} \tag{3.25}$$

and

$$\big|x_m^{(2j+1)}(t)\big| \geqslant \frac{H_{2j+2}T}{6}(t-\xi_{m,2j+1})^2, \quad t \in J,\ 0 \leqslant j \leqslant n-2,\ m \in \mathbb{N}, \tag{3.26}$$

where H_{2j} $(0 \leqslant j \leqslant n-1)$ is given by (3.11). Then (3.1) and (3.22)–(3.26) give

$$\begin{aligned}
&(0 \leqslant)\ f_m\big(t, x_m(t), \ldots, x_m^{(2n)}(t)\big) \\
&\quad \leqslant h\left(t, 2n+1+\sum_{j=0}^{2n}\big|x_m^{(j)}(t)\big|\right) + \sum_{j=0}^{n}\omega_{2j}\big(\big|x_m^{(2j)}(t)\big|\big) + \sum_{j=0}^{n-1}\omega_{2j+1}\big(\big|x_m^{(2j+1)}(t)\big|\big) \\
&\quad \leqslant h(t, 2n+1+VP) + \sum_{j=0}^{n-1}\omega_{2j}\big(H_{2j}t(T-t)\big) + \omega_{2n}\left(\frac{S}{1-r}t^{1-r}\right) \\
&\qquad + \sum_{j=0}^{n-2}\omega_{2j+1}\left(\frac{H_{2j+2}T}{6}(t-\xi_{m,2j+1})^2\right) \\
&\qquad + \omega_{2n-1}\left(\frac{S}{(1-r)(2-r)}|t-\xi_{m,2n-1}|^{2-r}\right)
\end{aligned}$$

for a.e. $t \in J$ and each $m \in \mathbb{N}$, where V is defined by (1.6). By (H_2), the functions $h(t, 2n+1+VP)$, $\omega_{2j}(H_{2j}t(T-t))$ $(0 \leqslant j \leqslant n-1)$ and $\omega_{2n}(St^{1-r}/(1-r))$ belong to $L_1(J)$, and so to prove the assertion of Lemma 3.4 it suffices to show that the sequences

$$\big\{\omega_{2n-1}\big(\Delta|t-\xi_{m,2n-1}|^{2-r}\big)\big\}, \quad \big\{\omega_{2j+1}\big(\Delta_j(t-\xi_{m,2j+1})^2\big)\big\}, \quad 0 \leqslant j \leqslant n-2,$$

are UAC on J where we set

$$\Delta = \frac{S}{(1-r)(2-r)}, \qquad \Delta_j = \frac{H_{2j+2}T}{6}.$$

Let $\{(a_k,b_k)\}_{k\in\mathbb{J}}$ be an at most countable set of mutually disjoint intervals $(a_k,b_k) \subset (0,T)$. Set

$$\mathbb{J}_m^{2j+1} = \{k\colon\ k \in \mathbb{J},\ (a_k, b_k) \subset (0, \xi_{m,2j+1})\},$$
$$\mathbb{K}_m^{2j+1} = \{k\colon\ k \in \mathbb{J},\ (a_k, b_k) \subset (\xi_{m,2j+1}, T)\}$$

for $m \in \mathbb{N}$ and $0 \leqslant j \leqslant n-1$. We first prove that $\{\omega_{2n-1}(\Delta|t-\xi_{m,2n-1}|^{2-r})\}$ is UAC on J. For $k \in \mathbb{J}_m^{2n-1}$ and $l \in \mathbb{K}_m^{2n-1}$ we have

$$\int_{a_k}^{b_k} \omega_{2n-1}\big(\Delta|t-\xi_{m,2n-1}|^{2-r}\big)\,dt = \int_{a_k}^{b_k} \omega_{2n-1}\big(\Delta(\xi_{m,2n-1}-t)^{2-r}\big)$$
$$= \frac{1}{\sqrt[2-r]{\Delta}} \int_{\sqrt[2-r]{\Delta}(\xi_{m,2n-1}-b_k)}^{\sqrt[2-r]{\Delta}(\xi_{m,2n-1}-a_k)} \omega_{2n-1}\big(s^{2-r}\big)\,ds,$$

$$\int_{a_l}^{b_l} \omega_{2n-1}\big(\Delta|t-\xi_{m,2n-1}|^{2-r}\big)\,dt = \int_{a_l}^{b_l} \omega_{2n-1}\big(\Delta(t-\xi_{m,2n-1})^{2-r}\big)$$
$$= \frac{1}{\sqrt[2-r]{\Delta}} \int_{\sqrt[2-r]{\Delta}(a_l-\xi_{m,2n-1})}^{\sqrt[2-r]{\Delta}(b_l-\xi_{m,2n-1})} \omega_{2n-1}\big(s^{2-r}\big)\,ds,$$

and if $\{k_0\} = \mathbb{J} \setminus (\mathbb{J}_m^{2n-1} \cup \mathbb{K}_m^{2n-1})$, that is $a_{k_0} < \xi_{m,2n-1} < b_{k_0}$, then

$$\int_{a_{k_0}}^{b_{k_0}} \omega_{2n-1}\big(\Delta|t-\xi_{m,2n-1}|^{2-r}\big)\,dt$$
$$= \frac{1}{\sqrt[2-r]{\Delta}} \left[\int_0^{\sqrt[2-r]{\Delta}(\xi_{m,2n-1}-a_{k_0})} \omega_{2n-1}\big(s^{2-r}\big)\,ds + \int_0^{\sqrt[2-r]{\Delta}(b_{k_0}-\xi_{m,2n-1})} \omega_{2n-1}\big(s^{2-r}\big)\,ds \right].$$

Hence

$$\sum_{k\in\mathbb{J}} \int_{a_k}^{b_k} \omega_{2n-1}\big(\Delta|t-\xi_{m,2n-1}|^{2-r}\big)\,dt$$
$$\leqslant \frac{1}{\sqrt[2-r]{\Delta}} \left[\int_{\mathcal{N}_m^1} \omega_{2n-1}\big(s^{2-r}\big)\,ds + \int_{\mathcal{N}_m^2} \omega_{2n-1}\big(s^{2-r}\big)\,ds \right]$$

where

$$\mathcal{N}_m^1 = \mathcal{E}_m^1 \cup \bigcup_{k\in\mathbb{J}_m^{2n-1}} \big(\sqrt[2-r]{\Delta}(\xi_{m,2n-1}-b_k),\ \sqrt[2-r]{\Delta}(\xi_{m,2n-1}-a_k)\big),$$
$$\mathcal{N}_m^2 = \mathcal{E}_m^2 \cup \bigcup_{l\in\mathbb{K}_m^{2n-1}} \big(\sqrt[2-r]{\Delta}(a_l-\xi_{m,2n-1}),\ \sqrt[2-r]{\Delta}(b_l-\xi_{m,2n-1})\big),$$

with

$$\mathcal{E}_m^1 = \begin{cases} \emptyset & \text{if } \mathbb{J} = \mathbb{J}_m^{2n-1} \cup \mathbb{K}_m^{2n-1}, \\ \bigl(0,\ \sqrt[2-r]{\Delta}(\xi_{m,2n-1} - a_{k_0})\bigr) & \text{if } \{k_0\} = \mathbb{J} \setminus \bigl(\mathbb{J}_m^{2n-1} \cup \mathbb{K}_m^{2n-1}\bigr), \end{cases}$$

$$\mathcal{E}_m^2 = \begin{cases} \emptyset & \text{if } \mathbb{J} = \mathbb{J}_m^{2n-1} \cup \mathbb{K}_m^{2n-1}, \\ \bigl(0,\ \sqrt[2-r]{\Delta}(b_{k_0} - \xi_{m,2n-1})\bigr) & \text{if } \{k_0\} = \mathbb{J} \setminus \bigl(\mathbb{J}_m^{2n-1} \cup \mathbb{K}_m^{2n-1}\bigr). \end{cases}$$

Since $\mu(\mathcal{N}_m^i) \leqslant \sqrt[2-r]{\Delta} \sum_{k \in \mathbb{J}}(b_k - a_k)$ $(i = 1, 2)$ and $\omega_{2n-1}(s^{2-r}) \in L_1([0,\ \sqrt[2-r]{\Delta}T])$ by (1.8), we see that $\{\omega_{2n-1}(\Delta|t - \xi_{m,2n-1}|^{2-r}\}$ is UAC on J.

Finally, we will show that $\{\omega_{2j+1}(\Delta_j(t - \xi_{m,2n-1})^2)\}$ $(0 \leqslant j \leqslant n-2)$ are UAC on J. Fix j, $0 \leqslant j \leqslant n-2$. We can proceed analogously to the proof of $\{\omega_{2n-1}(\Delta|t - \xi_{m,2n-1}|^{2-r})\}$ now with $r = 0$ to prove that

$$\sum_{k \in \mathbb{J}} \int_{a_k}^{b_k} \omega_{2j+1}\bigl(\Delta_j(t - \xi_{m,2n-1})^2\bigr)\,dt \leqslant \frac{1}{\sqrt[2-r]{\Delta_j}} \left[\ \int_{\mathcal{M}_m^1} \omega_{2j+1}\bigl(s^2\bigr)\,ds + \int_{\mathcal{M}_m^2} \omega_{2j+1}\bigl(s^2\bigr)\,ds \right]$$

where $\mu(\mathcal{M}_m^i) \leqslant \sqrt[2-r]{\Delta_j} \sum_{k \in \mathbb{J}}(b_k - a_k)$, $i = 1, 2$. Then from (see (1.7)) $\omega_{2j+1}(s^2) \in L_1([0,\ \sqrt[2-r]{\Delta_j}T])$ we deduce that $\{\omega_{2j+1}(\Delta_j(t - \xi_{m,2n-1})^2)\}$ is UAC on J. □

4. Existence result

Theorem 4.1. *Let assumptions* (H_1) *and* (H_2) *be satisfied. Then there exists a solution of BVP* (1.1), (1.2).

Proof. We will use Theorem 1.3 in this proof. Let the set $\mathcal{B}$ in the boundary conditions (1.10) be defined by $\mathcal{B} = \{x\colon\ x \in C^{(2n)}(J),\ x \text{ satisfies (1.2)}\}$. Consider the sequence of regular BVPs (1.9), (1.2) where f_m in (1.9) is defined in the beginning of Section 3.

Step 1. According to the definition $f_m \in \mathrm{Car}(J \times \mathbb{R}^{2n+1})$ $(m \in \mathbb{N})$ and f_m satisfies (1.13). Lemma 3.2 guarantees that BVP (1.9), (1.2) has a solution x_m for each $m \in \mathbb{N}$. Consider the sequence $\{x_m\}$. By Lemma 3.3, there is a positive constant P such that the inequalities (3.22) are satisfied and, in addition, (3.23)–(3.26) hold where H_{2j} $(0 \leqslant j \leqslant n-1)$ is given by (3.11) and $\xi_{m,2j+1} \in (0, T)$ $(m \in \mathbb{N}, 0 \leqslant j \leqslant n-1)$ is the unique zero of $x_m^{(2j+1)}$. Define

$$\Omega = \bigl\{x\colon\ x \in C^{2n}(J),\ \|x\|_{C^{2m}} < r^*\bigr\},$$

where $r^* = P \sum_{j=0}^{2n} T^{2n-j}$. Then $\{x_m\} \subset \Omega$. Finally, $\{f_m(t, x_m(t), \dots, x_m^{(2n)}(t))\}$ is UAC on J by Lemma 3.4. Therefore the assumptions (a), (b) of Theorem 1.3 are fulfilled. Hence its assertion (A) is true, that is there exists $x \in \overline{\Omega}$ and a subsequence $\{x_{k_m}\}$ of $\{x_m\}$ such that $\lim_{m \to \infty} x_{k_m} = x$ in $C^{2n}(J)$. Also without restriction of generality we can assume that $\{\xi_{k_m,2j+1}\}_{m \in \mathbb{N}}$ is convergent for $0 \leqslant j \leqslant n-1$ and $\lim_{m \to \infty} \xi_{k_m,2j+1} = \xi_{2j+1}$.

Step 2. Now, we describe the set of all zeros of the functions $x^{(i)}$, $0 \leqslant i \leqslant 2n$. Letting $m \to \infty$ in (3.23)–(3.26) (where we put k_m instead of m), we get

$$\left|x^{(2n)}(t)\right| \geqslant \frac{S}{1-r} t^{1-r}, \quad \left|x^{(2n-1)}(t)\right| \geqslant \frac{S}{(1-r)(2-r)} |t - \xi_{2n-1}|^{2-r}, \quad t \in J,$$
$$\left|x^{(2j)}(t)\right| \geqslant H_{2j} t(T-t), \quad t \in J,\ 0 \leqslant j \leqslant n-1$$

and

$$\left|x^{(2j+1)}(t)\right| \geqslant \frac{H_{2j+2}T}{6} (t - \xi_{2j+1})^2, \quad t \in J,\ 0 \leqslant j \leqslant n-2.$$

These inequalities imply that the set of all zeros of $x^{(i)}$, $0 \leqslant i \leqslant 2n$, is finite and hence, by Theorem 1.3, x is a solution of BVP (1.1), (1.2). □

References

[1] R.P. Agarwal, Boundary Value Problems for Higher Order Differential Equations, World Scientific, Singapore, 1986.
[2] R.P. Agarwal, M. Meehan, D. O'Regan, Fixed Point Theory and Applications, Cambridge Univ. Press, 2001.
[3] R.P. Agarwal, P.J.Y. Wong, On Lidstone polynomials and boundary value problems, Comput. Math. Appl. 17 (1989) 1397–1421.
[4] R.P. Agarwal, D. O'Regan, I. Rachůnková, S. Staněk, Two point higher order BVPs with singularities in phase variables, Comput. Math. Appl., in press.
[5] A. Granas, R. Guenther, J. Lee, Nonlinear Boundary Value Problems for Ordinary Differential Equations, in: Dissertationes Math., vol. 244, Polish Acad. Sci., Warszawa, 1985.
[6] A. Granas, R. Guenther, J. Lee, Some general existence principles in the Carathéodory theory of nonlinear differential systems, J. Math. Pures Appl. 70 (1991) 153–196.
[7] P.K. Palamides, Positive solutions for higher order Lidstone boundary value problems. A new approach via Sperner's lemma, Comput. Math. Appl., in press.
[8] I. Rachůnková, S. Staněk, General existence principle for singular BVPs and its applications, Georgian Math. J., in press.
[9] P.J.Y. Wong, R.P. Agarwal, Error Inequalities in Polynomial Interpolation and their Applications, Kluwer Academic, Dordrecht, 1993.
[10] P.J.Y. Wong, R.P. Agarwal, Eigenvalues of Lidstone boundary value problems, Appl. Math. Comput. 104 (1999) 15–31.